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INTERACTION OF TWO EQUAL COPLANAR SQUARE CRACKS IN THREE-DIMENSIONAL ELASTICITY

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Abstract—Using the Kelvin–Somigliana formulae, the problem of two coplanar square cracks in a three-dimensional infinite isotropic elastic body is reduced to solving three two-dimensional singular boundary integral equations. The numerical method of quadratic elements is proposed for the above-mentioned equations. The stress intensity factors of Mode I, Mode II and Mode III are numerically calculated for a few examples. The obtained results explain the effect of one crack on another crack and also show the effectiveness of the present method.

1. INTRODUCTION

The boundary element methods have been widely applied to the stress intensity factor computation of cracked bodies. At present, there are two types of the boundary integral equations in crack analysis. One is the so-called direct formulation [e.g. Tanaka and Itoh (1986)] in which the cracked body is subdivided along the crack planes. Another is the indirect formulation [e.g. Bui (1977) and Weaver (1977)] in which the stress conditions on crack faces are satisfied. Each formulation has advantages and disadvantages. The present authors prefer the indirect formulation because it is more economical. In this paper, a new indirect formulation of two coplanar square cracks in a three-dimensional infinite isotropic elastic body is derived from the Kelvin–Somigliana formulae [see, e.g. Danson (1983)]. Three types of crack elements are proposed for the numerical treatment of the formulation. The stress intensity factors of Mode I, Mode II and Mode III are numerically calculated for a few typical examples. The obtained results explain the effect of one crack on another crack and also show the effectiveness of the present method.

2. BOUNDARY INTEGRAL EQUATIONS

Consider the problem of an infinite isotropic elastic body with two coplanar square cracks Γ_1 and Γ_2 located on the coordinate plane Ox_1x_2 in the Cartesian coordinate system $Ox_1x_2x_3$ (see Fig. 1). And without loss of generality, suppose the stress at infinity and the body force are equal to zero.

From the Kelvin–Somigliana formulae [see, e.g. Danson (1983)] we have

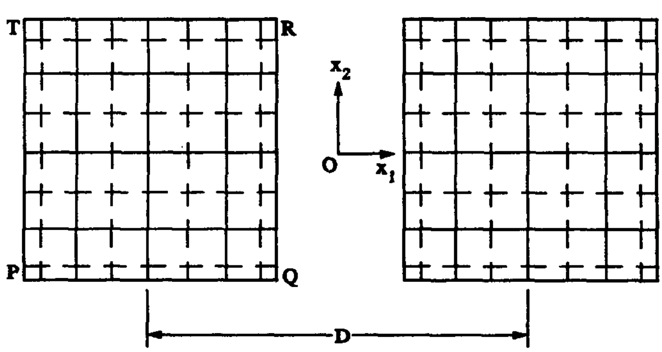

Fig. 1. Two coplanar square cracks ($|PQ| = 2A$) in an infinite body.

$$u_i(x) = -\int_\Gamma T_{ij}(x,y)\Delta u_j(y)\,\mathrm{d}s(y) + \int_\Gamma U_{ij}(x,y)\Sigma t_j(y)\,\mathrm{d}s(y)$$
$$[y = (y_1, y_2, 0) \in \Gamma = \Gamma_1 \bigcup \Gamma_2, \quad x = (x_1, x_2, x_3) \notin \Gamma], \tag{1}$$

where

$$U_{ij}(x,y) = \frac{1}{8\pi\mu}\left[\delta_{ij} r_{,kk} - \frac{r_{,ij}}{2(1-v)}\right], \quad T_{ij}(x,y) = -\delta_{3j}\lambda U_{ik,k} - \mu(U_{ij,3} + U_{i3,j}),$$
$$\Delta u_j(y) = u_j(y^+) - u_j(y^-), \quad \Sigma t_j(y) = \sigma_{3j}(y^-) - \sigma_{3j}(y^+), \quad y^\pm = (y_1, y_2, \pm 0). \tag{2}$$

u_i and σ_{ij} are the displacement and the stress, respectively, λ and μ are Lamé's constants and v is Poisson's ratio.

Substituting eqn (1) into the stress–displacement relations and letting the internal stresses satisfy the boundary stressed conditions on the upper and lower surfaces of two cracks, we can obtain the three following two-dimensional singular boundary integral equations about the crack dislocations $F_i(y)$ $(i = 1, 2, 3)$:

$$\frac{1}{2\pi}\int_\Gamma \frac{1}{r^2} r_{,\beta}[F_{3,\beta}(y) - \omega\Sigma t_\beta(y)]\,\mathrm{d}s(y) = P_3(x),$$
$$\frac{1}{2\pi}\int_\Gamma \frac{1}{r^2}\{r_{,1}[W_2(y) + \omega\Sigma t_3(y)] - r_{,2} W_3(y)\}\,\mathrm{d}s(y) = P_1(x),$$
$$\frac{1}{2\pi}\int_\Gamma \frac{1}{r^2}\{r_{,2}[W_2(y) + \omega\Sigma t_3(y)] + r_{,1} W_3(y)\}\,\mathrm{d}s(y) = P_2(x)$$
$$(r = |y - x|, y = (y_1, y_2), x = (x_1, x_2) \in \Gamma), \tag{3}$$

where

$$\omega = \frac{1-2v}{4(1-v)}, \quad P_i(x) = \tfrac{1}{2}[\sigma_{3i}(x^+) + \sigma_{3i}(x^-)], \tag{4}$$

$$W_2(y) = F_{1,1}(y) + F_{2,2}(y), W_3(y) = (1-v)[F_{2,1}(y) - F_{1,2}(y)], \quad F_i(y) = \frac{\mu}{2(1-v)}\Delta u_i(y). \tag{5}$$

The dummy index β is only summed from 1 to 2.

The crack dislocations $F_i(y)$ $(i = 1, 2, 3)$ must also satisfy the single valued conditions of displacement along the crack edges $\partial\Gamma = \partial\Gamma_1 \cup \partial\Gamma_2$:

$$F_i(z) = 0, \quad (i = 1, 2, 3; z \in \partial\Gamma). \tag{6}$$

3. STRESS INTENSITY FACTORS

The stress intensity factors K_{I} (mode I), K_{II} (mode II) and K_{III} (mode III) at a point z on the crack edge $\partial\Gamma$ are defined as

$$K_{\mathrm{I}}(z) = \lim_{r\to 0}\{\sqrt{2\pi r}\sigma_{33}(x)\}, \quad K_{\mathrm{II}}(z) = \lim_{r\to 0}\{\sqrt{2\pi r}\sigma_{3n}(x)\},$$
$$K_{\mathrm{III}}(z) = \lim_{r\to 0}\{\sqrt{2\pi r}\sigma_{3t}(x)\}, (r = |x - z|, x = (x_1, x_2, 0) \notin \bar{\Gamma}), \tag{7}$$

where $\sigma_{33}(x)$, $\sigma_{3n}(x)$ and $\sigma_{3t}(x)$ are the stress components with respect to the curvilinear coordinates (n, t, x_3) in the vicinity of the point z, n and t, respectively, are the outward normal and the tangent to $\partial\Gamma$.

From the results of Bui (1977), we can obtain the limiting formulae of stress intensity factors as follows:

$$K_{\rm I}(z) = \lim_{y\to z}\left\{\left(\frac{\pi}{2d}\right)^{1/2} F_3(y)\right\},$$

$$K_{\rm II}(z) = \lim_{y\to z}\left\{\left(\frac{\pi}{2d}\right)^{1/2} [n_1(z)F_1(y)+n_2(z)F_2(y)]\right\},$$

$$K_{\rm III}(z) = \lim_{y\to z}\left\{\left(\frac{\pi}{2d}\right)^{1/2} (1-\nu)[-n_2(z)F_1(y)+n_1(z)F_2(y)]\right\} \quad (z\in\partial\Gamma), \tag{8}$$

where d denotes the distance of the point y to the tangent zt and $n_i(z)$ the unit outward normal.

Once the boundary integral equations (3) about $F_i(y)$ are solved, the stress intensity factors can be determined directly from (8).

4. CRACK ELEMENTS

In general, an exact analysis of the boundary integral equations (3) is very difficult. Thus, to obtain its numerical solution, two cracks Γ are divided into a series of quadratic elements with 9 nodes (see Fig. 1), that is, $\Gamma = U\Gamma_e$. Each element Γ_e is mapped onto a square ($|\xi| \leqslant 1$ and $|\eta| \leqslant 1$) in local coordinates (ξ, η) (see Fig. 2). For the interpolation functions of the unknown dislocations $F_i(x)$, there are three types of crack elements:

(1) Crack interior element. Element nodes are the internal points of the cracks Γ. In this case, $F_i(x)$ can be interpolated by using the ordinary isoparametric element.

(2) Crack edge element. Element nodes 2, 6 and 3 belong to the crack edge $\partial\Gamma$. On this element, $F_i(x)$ can be approximated by

$$F_i(x) = \sqrt{(1-\xi)/2}\sum_{k=1}^{6} G^e_{ik}T_k(\xi,\eta) \quad (|\xi|\leqslant 1, |\eta|\leqslant 1), \tag{9}$$

where

$$(G^e_{i1}, G^e_{i2}, G^e_{i3}, G^e_{i4}, G^e_{i5}, G^e_{i6}) = (F^e_{i1}, F^e_{i5}, F^e_{i9}, F^e_{i7}, F^e_{i4}, F^e_{i8}),$$

$$T_1 = \tfrac{1}{2}\xi(1-\eta)\eta,\quad T_2 = -\frac{1}{\sqrt{2}}(1+\xi)(1-\eta)\eta,\quad T_3 = \sqrt{2}(1+\xi)(1-\eta^2),$$

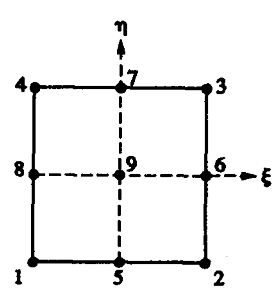

Fig. 2. Element Γ_e.

$$T_4 = \frac{1}{\sqrt{2}}(1+\xi)\eta(1+\eta), \quad T_5 = -\tfrac{1}{2}\xi\eta(1+\eta), \quad T_6 = -\xi(1-\eta^2). \tag{10}$$

F^e_{ik} is the nodal value of $F_i(x)$.

(3) Crack corner element. Element nodes 2, 6, 3, 7 and 4 belong to $\partial\Gamma$ and the node 3 is a corner point of $\partial\Gamma$. For a crack corner element, $F_i(x)$ can be approximately expressed as

$$F_i(x) = \sqrt{(1-\xi)(1-\eta)/4}\sum_{k=1}^{4} G^e_{ik}T_k(\xi,\eta) \quad (|\xi| \leqslant 1, |\eta| \leqslant 1), \tag{11}$$

where

$$(G^e_{i1}, G^e_{i2}, G^e_{i3}, G^e_{i4}) = (F^e_{i1}, F^e_{i5}, F^e_{i9}, F^e_{i8}),$$

$$T_1 = \xi\eta, \quad T_2 = -\sqrt{2}(1+\xi)\eta, \quad T_3 = 2(1+\xi)(1+\eta), \quad T_4 = -\sqrt{2}\xi(1+\eta). \tag{12}$$

Using the above-mentioned crack elements and interpolation functions, eqn (3) can be discretized into a set of linear algebraic equations about the nodal values F_{ik} of $F_i(x)$. Therefore, the numerical solutions of eqn (3) and K-factors can be obtained.

5. EXAMPLES

For linear fracture analysis, one takes interest in the stress intensity factors. Thus, in this section, we will evaluate the stress intensity factors of a few typical examples by using the suggested method of this paper. Two square cracks are divided into 32 elements with 98 internal nodes (see Fig. 1). When the load is a uniform tension p perpendicular to the cracks, the numerical dimensionless stress intensity factors $K_1 = K_I/p\sqrt{A}$ (mode I) are shown in Figs 3, 4. Figures 5, 6 give $K_2 = K_{II}/q\sqrt{A}$ (mode II) and $K_3 = K_{III}/q\sqrt{A}$ (mode

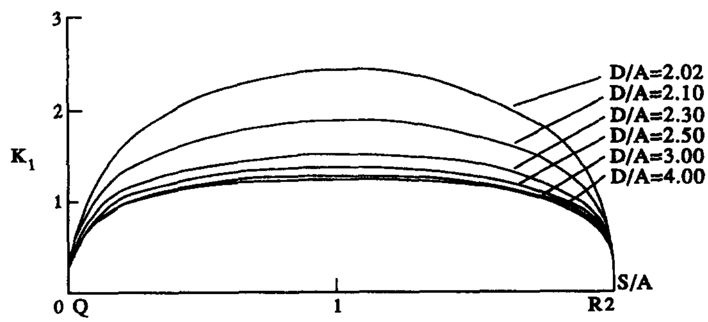

Fig. 3. $K_1 = K_I/p\sqrt{A}$ along side QR of two square cracks.

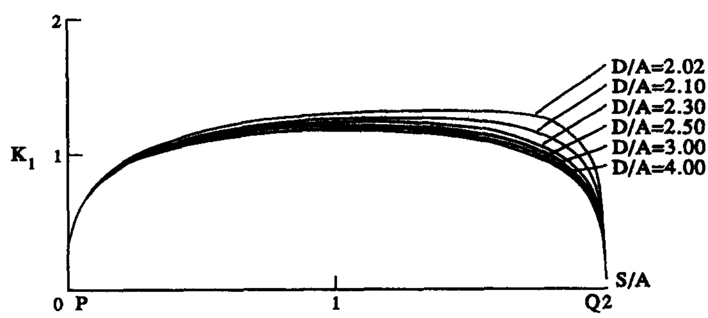

Fig. 4. $K_1 = K_I/p\sqrt{A}$ along side PQ of two square cracks.

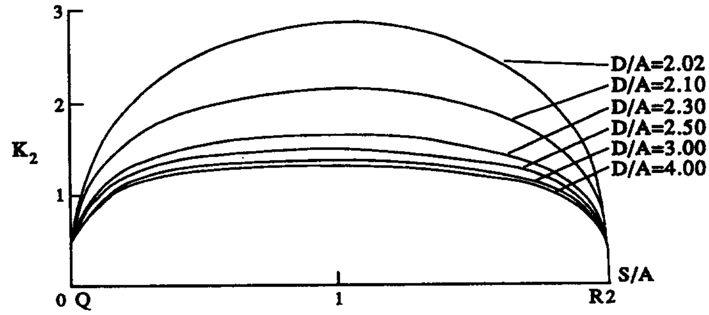

Fig. 5. $K_2 = K_{II}/q\sqrt{A}$ along side QR of two square cracks.

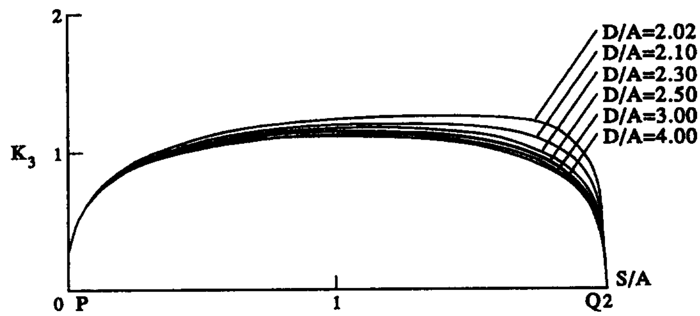

Fig. 6. $K_3 = K_{III}/q\sqrt{A}$ along side PQ of two square cracks.

III) in the case of a uniform shear q parallel to the x_1 axis. Obviously, the smaller the dimensionless parameter D/A, the greater is the effect of one crack on another crack. When $D/A > 4$, this effect can be omitted and the results are almost the same as those of a single square crack [e.g. Wang (1991)].

6. CONCLUSION

The new boundary integral equations (3) of two equal coplanar square cracks are given, in which $W_2(y)$ and $W_3(y)$ are the dislocation densities of two variables. The above equations can be also used in analysis of many coplanar cracks of arbitrary shape in three-dimensional elasticity. Three types of crack elements are proposed for the numerical treatment of eqns (3). The numerical results of a few examples show the effect of one crack on another crack and also verify the validity and effectiveness of the present method, which are not seen in the literature.

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REFERENCES

Bui, H. D. (1977). An integral equations method for solving the problem of a plane crack of arbitrary shape. *J. Mech. Phys. Solids* **25**, 29–39.

Danson, D. (1983). Linear isotropic elasticity with body forces. In *Progress in Boundary Element Methods* (Edited by C. A. Brebbia), Vol. 2, pp. 101–135. Pentech Press, London.

Tanaka, M. and Itoh, H. (1986). Boundary element modeling of crack singularities in 3-D elastostatics. In *Boundary Elements* (Edited by Q. H. Du), pp. 275–285. Pergamon Press, Oxford.

Wang, Y. B. (1991). A boundary integral equations method of three-dimensional crack problems. *Proceedings of Third Conference of Boundary Element Methods in Engineering*, Wuhan, China, 23–26 December. pp. 160–165 (in Chinese).

Weaver, J. (1977). Three-dimensional crack analysis. *Int. J. Solids Structures* **13**, 321–330.

APPENDIX

The function $r = |y-x|$ in the first formula of (2) satisfies the biharmonic equation:

$$\Delta^2 r = 0 \quad (r \neq 0), \tag{A1}$$

where Δ is a three-dimensional harmonic operator.

Substituting (1) into the stress–displacement relations:

$$\sigma_{ij} = \lambda \delta_{ij} u_{k,k} + \mu(u_{i,j} + u_{j,i}), \tag{A2}$$

and together using (A1) and integration by parts, we can obtain three internal stress components:

$$\sigma_{33}(x) = \int_\Gamma T_{j3}(x,y)\Sigma t_j(y)\,\mathrm{d}s(y) - \frac{1}{2\pi}\int_\Gamma [F_{,\beta}(y)\Delta_2 r_{,\beta} - W_2(y)\Delta_2 r_{,3}]\,\mathrm{d}s(y),$$

$$\sigma_{31}(x) = \int_\Gamma T_{j1}(x,y)\Sigma t_j(y)\,\mathrm{d}s(y) + \frac{1}{2\pi}\int_\Gamma \left[F_{,1}(y)\Delta_2 r_{,3} - W_2(y) r_{,133} + W_3(y)\left(\frac{1}{r}\right)_{,2}\right]\mathrm{d}s(y),$$

$$\sigma_{32}(x) = \int_\Gamma T_{j2}(x,y)\Sigma t_j(y)\,\mathrm{d}s(y) + \frac{1}{2\pi}\int_\Gamma \left[F_{,2}(y)\Delta_2 r_{,3} - W_2(y) r_{,233} - W_3(y)\left(\frac{1}{r}\right)_{,1}\right]\mathrm{d}s(y)$$

$$(\Delta_2 \text{ is a two-dimensional Laplacian}). \tag{A3}$$

Now letting (A3) satisfy the boundary stressed conditions on the upper and lower crack surfaces of two cracks, we can obtain six boundary integral equations:

$$\sigma_{33}(x^{\pm}) = \frac{1}{2\pi}\int_\Gamma \frac{1}{r^2} r_{,\beta}[F_{,\beta}(y) - \omega\Sigma t_\beta(y)]\,\mathrm{d}s(y) \mp \tfrac{1}{2}\Sigma t_3(x),$$

$$\sigma_{31}(x^{\pm}) = \frac{1}{2\pi}\int_\Gamma \frac{1}{r^2}\{r_{,1}[W_2(y) + \omega\Sigma t_3(y)] - r_{,2} W_3(y)\}\,\mathrm{d}s(y) \mp \tfrac{1}{2}\Sigma t_1(x),$$

$$\sigma_{32}(x^{\pm}) = \frac{1}{2\pi}\int_\Gamma \frac{1}{r^2}\{r_{,2}[W_2(y) + \omega\Sigma t_3(y)] + r_{,1} W_3(y)\}\,\mathrm{d}s(y) \mp \tfrac{1}{2}\Sigma t_2(x)(x^{\pm} = (x_1, x_2, \pm 0),$$

$$x = (x_1, x_2, 0) \in \Gamma), \tag{A4}$$

in which only three equations (3) are independent.